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Multivariate weak spline function space ☆

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Abstract

In this paper, B-net method for studying multivariate weak spline is discussed and the dimension of $W_2^1(I_1\Delta^{(1)})$ is presented. Several B splines on type-1 triangulation are also obtained.

Keywords: Spline; Multivariate weak spline; Dimension; B spline

1. Introduction

It is well known that the ordinary multivariate spline is defined by piecewise polynomials with certain smoothness on a domain while the multivariate weak spline is defined by a piecewise polynomial which smooths only on a set of discrete points. The multivariate weak spline is important in finite element and CAGD and it is defined and discussed in [8]. As we know, B-net method is an important tool for studying multivariate spline. In this paper, we will discuss how to study multivariate weak spline using B-net method. We use B-net as a tool in deriving the dimension of $W_2^1(I_1\Delta)$. Several B splines on type-1 triangulation are also presented. To help make this paper self-contained, the remainder of this section contains a brief statement of the relevant definitions.

In this paper, $\mathbf{P_k(x,y)}$ denotes the collection of all bivariate polynomials of real coefficients with total degree k, and $\mathbf{P_k(x)}$ denotes the collection of all univariate polynomials of real coefficients with total degree k. $D^mP(x_i,y_i)$ denotes the collection of $\partial^m P(x,y)/\partial^t x\partial^{m-t}y|_{(x_i,y_i)}$, $0\leqslant t\leqslant m$.

Definition 1. Let l be a line segment and the point set $S=\{(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)\}$ lie on l. If $|S|:=\mathrm{card}(S)$ is limited and each point of S is an interior point of the line segment l, then S is called an appointed point set and every point in S is called an appointed point.

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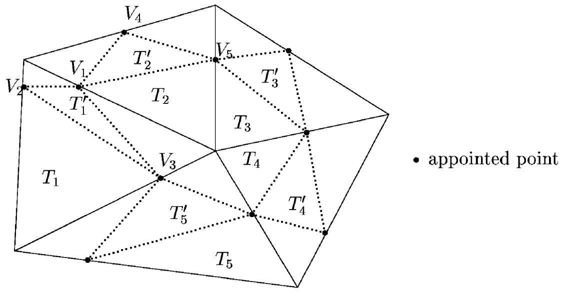

Fig. 1. An appointed point partition.

Let D be a domain in $\Re^2, \Delta$ be a partition of the domain D consisting of finite straight lines or line segments, and an appointed point set is given on each inner edge. This partition is called the *appointed point partition* which is denoted by $I\Delta$. Let D_i, $i=1,\dots,T$, be all of the cells of Δ and S_j, $j=1,\dots,L$ be all of the appointed point sets of $I\Delta$. Denote by $C^{\mu}_{(0)}(S)$ the set of functions with smoothness μ in each point of the given point set S. For integer $k > \mu \geqslant 0$,

$$W^{\mu}_{k}(I\Delta) = \left\{ W(x,y) \in C^{\mu}_{(0)}(S) \,|\, W(x,y)|_{D_i} \in \mathbf{P_k(x,y)},\ \forall D_i,\ S = \bigcup_{j} S_{\Gamma_j} \right\} \tag{1}$$

is called a *multivariate weak spline space with degree k and smoothness* μ, where $W(x,y)|_D$ denotes the restriction of $W(x,y)$ on D.

If $S(x,y) \in W^{\mu}_{k}(I\Delta) \cap S^{\mu}_{k}(\Delta)$, then $S(x,y)$ is called a singular multivariate weak spline, where $S^{\mu}_{k}(\Delta)$ is an ordinary multivariate spline space. In this paper, we only consider a simple but practical case, i.e. the cardinality of appointed point set on each grid segment is 1 and Δ is a triangulation of a simply connected domain $D \subset \Re^2$. Denote by $I_1\Delta$ the appointed point triangulation and by $W^{\mu}_{k}(I_1\Delta)$ the multivariate weak spline space over $I_1\Delta$.

2. B-net method for studying multivariate weak spline

We begin with introducing some additional notations. Let Δ be a triangulation of a simply connected domain $D \subset \Re^2$. Let N be the number of triangles, E_{I} the number of interior edges, E_{B} the number of boundary edges, V the number of vertices and V_{I} the number of interior vertices.

The cardinality of appointed point set on each grid segment is 1. Let the set of all appointed points be S. For each triangle T_i, there are three appointed points $p_{i1}, p_{i2}, p_{i3} \in T_i$. See Fig. 1, connecting p_{i1}, p_{i2}, and p_{i3}, we get a new triangle which is denoted as T_i'.

Obviously, for each weak spline $W(x,y) \in W^{\mu}_{k}(I_1\Delta)$, $W(x,y)|_{T_i} \equiv W(x,y)|_{T_i'}$, $i=1,\dots,N$. Let

$$p_i(x,y) = W(x,y)|_{T_i'}, \quad \text{for } (x,y) \in T_i',\ i=1,\dots,N, \tag{2}$$

where, p_i is a polynomial of degree k. Each of these polynomials can be written in Bézier–Bernstein form as

$$p_i(\alpha,\beta,\gamma)=\sum_{l+h+d=k} c^i_{lhd}\frac{k!}{l!h!d!}\alpha^l\beta^h\gamma^d, \tag{3}$$

where (α,β,γ) are the barycentric coordinates of a point (x,y) in the triangle T_i', c^i_{lhd} is called Bézier ordinate.

With each Bézier ordinate c^i_{lhd} we associate a domain point

$$P^i_{lhd}=(lV_1+hV_2+dV_3)/k,$$

where V_1, V_2, and V_3 denote the vertices of the triangle T_i'. We omit the superscript i whenever this will cause no confusion. The set of all domain points is denoted by $\mathscr{B}_k(\varDelta)$. The Bézier net of a function $w(x,y)\in W^0_k(I_1\varDelta)$ is the set of points $(P_{lhd},c_{lhd})\in\mathfrak{R}^3$. A given Bézier net uniquely defines a function in $W^0_k(I_1\varDelta)$. We say that the point P_{lhd} is of distance $k-l$ from vertex V_1 (with similar definitions for the other two vertices). The *ring of order p* around the vertex v is

$$R_p(v)=\{\text{points which have distance } p \text{ from } v\}$$

and the *disk of order p* around v is

$$D_p(v)=\bigcup_{j=0}^{p}R_j(v).$$

Now we discuss the smoothing condition. See Fig. 1, let T_1' denote a triangle with vertices V_1,V_2,V_3, and let T_2' denote a triangle with vertices V_1,V_4,V_5. We write a polynomial p_1 on T_1' in its Bernstein–Béier form (3), and a polynomial p_2 on T_2' similarly. Let $(\alpha_4,\beta_4,\gamma_4)$ and $(\alpha_5,\beta_5,\gamma_5)$ denote the barycentric coordinates of V_4 and V_5 with respect to T_1'. Then, in order for p_1 and p_2 to be joined μ times differentiably across the common vertices V_1, we must have [5]

$$\varDelta^{v_1}_{1,0}\varDelta^{v_2}_{2,0}c^2_{l00}=(\beta_4\varDelta_{1,0}+\gamma_4\varDelta_{2,0})^{v_1}(\beta_5\varDelta_{1,0}+\gamma_5\varDelta_{2,0})^{v_2}c^1_{l00}, \tag{4}$$

where $v_1+v_2=g$, $g=0,\dots,\mu$, $l=k-g$, $\varDelta_{i,j}c_{(\alpha,\beta,\gamma)}=c_{s_i(\alpha,\beta,\gamma)}-c_{s_j(\alpha,\beta,\gamma)}$, $s_1(\alpha,\beta,\gamma)=(\alpha+1,\beta,\gamma)$, $s_2(\alpha,\beta,\gamma)=(\alpha,\beta+1,\gamma)$, $s_3(\alpha,\beta,\gamma)=(\alpha,\beta,\gamma+1)$.

Especially, for $\mu=1$, (4) shows that the domain points in the disk of order 1 around V_1 are coplanar.

Given a point $t\in\mathscr{B}_k(\varDelta)$, let λ_t be the linear functional defined on $W^0_k(I_1\varDelta)$ with the property that

$$\lambda_t w=\text{Bézier ordinate of } w \text{ associated with } t.$$

A set $P\subset\mathscr{B}_k(\varDelta)$ is said to be a *determining set* for $W^\mu_k(I_1\varDelta)$ if

$$\forall w\in W^\mu_k(I_1\varDelta)\colon \lambda_t w=0,\quad \forall t\in P\Rightarrow w=0. \tag{5}$$

3. The dimension of space $W^1_2(I_1\varDelta)$

Denote by $St(v)$ the collection of cells in $\varDelta$ sharing v as a common vertex. $St(v)$ is called *a standard cell*. Denote by $W^\mu_k(St(v))$ the multivariate weak spline space over $St(v)$. It is difficult to obtain

$\dim W_k^\mu(St(v))$ and $\dim W_k^\mu(I_1\Delta)$. In [8], the dimension of $W_k^\mu(St(v))$ as well as $W_k^\mu(I_1\Delta)$, $k \geqslant 2\mu+1$ is presented. In this section, we will discuss $\dim W_2^1(St(v))$ and $\dim W_2^1(I_1\Delta)$. Let us first introduce some lemmas.

Lemma 1. *If P determines $W_k^\mu(I_1\Delta)$, then* $\dim W_k^\mu(I_1\Delta) \leqslant |P|$.

Proof. Let $B_1,\ldots,B_n$ be a basis for $W_k^\mu(I_1\Delta)$. Suppose $|P| < n$. Obviously, there exists a nontrivial solution of the system

$$\lambda_t\left(\sum_{j=1}^n c_j B_j\right) = 0, \quad t \in P.$$

But then the nontrivial weak spline $w = \sum_{j=1}^n c_j B_j$ contradicts our assumption that P determines $W_k^\mu(I_1\Delta)$. We have our conclusion. □

The following lemma gives a lower bound on the dimension of $W_k^\mu(I_1\Delta)$.

Lemma 2. *Let $I_1\Delta$ be an appointed point partition. Then* $\dim W_k^\mu(I_1\Delta) \geqslant N\binom{k+2}{2} - E_{\mathrm{I}}\binom{\mu+2}{2}$.

Proof. According to the definition of multivariate weak spline, we know $w(x,y)|_{T_i'} \in \mathbf{P_k}(\mathbf{x},\mathbf{y})$, where $w(x,y) \in W_k^\mu(I_1\Delta)$. Let $p_i(x,y) = w(x,y)|_{T_i'}$ and $p_i(x,y) = \sum_{m+n\leqslant k} a_{mn}^{(i)} x^m y^n$. Suppose $T_{i_l}, 1 \leqslant l \leqslant 3$ share an interior edge with T_i. Denote the appointed point shared by T_i and T_{i_m} as $(x_{i_m}, y_{i_m}), 1 \leqslant m \leqslant l$. We have

$$D^h p_i(x_{i_m}, y_{i_m}) = D^h p_{i_m}(x_{i_m}, y_{i_m}), \quad h = 0,\ldots,\mu, \;\; i = 1,\ldots,N. \tag{6}$$

Denote the solution space of system (6) as H. Obviously, $\dim W_k^\mu(I_1\Delta) = \dim H$. There are $N\binom{k+2}{2}$ unknown variables and $E_{\mathrm{I}}\binom{\mu+2}{2}$ equations in system (6). Therefore, $\dim H \geqslant N\binom{k+2}{2} - E_{\mathrm{I}}\binom{\mu+2}{2}$, i.e. $\dim W_k^\mu(I_1\Delta) \geqslant N\binom{k+2}{2} - E_{\mathrm{I}}\binom{\mu+2}{2}$. □

Lemma 3. *If $M = 2$, then* $\dim W_2^1(St(v)) = 7$. *If $M \geqslant 3$, then* $\dim W_2^1(St(v)) = 3M$, *where, M is the number of edges.*

Proof. If $M = 2$, it is easy to prove the result. Thus, we only consider $M \geqslant 3$. Label grid segment. The cell between the ith grid segment and the $(i+1)$th grid segment is called the ith cell, where $1 \leqslant i \leqslant M-1$ and the cell between the Mth grid segment and 1st grid segment is called the Mth cell. Denote by (x_i, y_i) the appointed point that lies in the ith grid segment. Without loss of generality, we assume that $(x_1, y_1) = (0,0)$, $x_M \neq 0$, $y_M \neq 0$, $x_{M-1} \neq 0$, and $y_{M-1} \neq 0$. Suppose that the polynomial defined on the ith cell is $p_i(x,y) = \sum_{n+m\leqslant 2} a_{nm}^{(i)} x^n y^m$. Let $u_{nm}^{(i)} = \partial^{n+m} W(x_i, y_i)/\partial x^n \partial y^m$, $n+m \leqslant 1$, where $W(x,y)$ is the multivariate weak spline concerning the vertex v. According to the definition of $W_2^1(St(v))$, $u_{nm}^{(i)}$ and $a_{nm}^{(i)}$ are unknown variables, we get the following coefficient matrix:

$$A=\begin{pmatrix} A_1 & 0 & 0 & 0 & -E & 0 & 0 & 0 & 0 \\ A_2 & 0 & 0 & 0 & 0 & -E & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & 0 & -E & 0 & 0 & 0 \\ 0 & A_3 & 0 & 0 & 0 & 0 & -E & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 & -E \\ 0 & 0 & 0 & A_M & 0 & 0 & 0 & 0 & -E \\ 0 & 0 & 0 & A_1 & -E & 0 & 0 & 0 & 0 \end{pmatrix},$$

where E is the identity matrix of order 3, each A_i is a matrix of order 3×6. The row vector of A_i is $\partial^{n+m}\mathbf{b}(\mathbf{x_i},\mathbf{y_i})/\partial x^n\partial y^m$, $n+m\leqslant 1$, where $\mathbf{b}(\mathbf{x},\mathbf{y})=(x^2,xy,y^2,x,y,1)$. It is clear that A_i is a matrix with full row rank. After a series of transformations, the matrix A becomes:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -E & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -E \\ 0 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_M & 0 & 0 & 0 & 0 \\ 0 & A_1 & A_1 & A_1 & A_1 & 0 & 0 & 0 & \end{pmatrix}.$$

Obviously, if $B=\begin{pmatrix} A_{M-1} & 0 \\ 0 & A_M \\ A_1 & A_1 \end{pmatrix}$ is of full row rank, then the matrix A is also of full row rank.

Now we consider

$$B=\begin{pmatrix} x_{M-1}^2 & x_{M-1}y_{M-1} & y_{M-1}^2 & x_{M-1} & y_{M-1} & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 2x_{M-1} & y_{M-1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & x_{M-1} & 2y_{M-1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & x_M^2 & x_My_M & y_M^2 & x_M & y_M & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & 2x_M & y_M & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_M & 2y_M & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Because of $x_M \neq 0$, $y_M \neq 0$, $x_{M-1} \neq 0$, $y_{M-1} \neq 0$, B is a matrix with full row rank. Thus, the number of unknown variables is $9M$ and the rank of coefficient matrix is $6M$. So $\dim W_2^1(St(v)) = 3M$. $\square$

Lemma 4. *Suppose $Q = \dim W_k^{\mu}(I_1\Delta)$. Then there exists a subset $P \subset \mathscr{B}_k(\Delta)$ with $|P| = Q$ such that* (5) *holds.*

Proof. Suppose that no set with the desired properties exists. We call this hypothesis H. Since $Q = \dim W_k^{\mu}(I_1\Delta)$, we shall construct $Q+1$ linearly independent elements in $W_k^{\mu}(I_1\Delta)$, thus providing a contradiction to the fact that Q is the dimension of $W_k^{\mu}(I_1\Delta)$. Let w_1 be the weak spline in $W_k^{\mu}(I_1\Delta)$ with all Bézier ordinates equal to 1. Let t_1 be a point in $\mathscr{B}_k(\Delta)$, and let P_1 be a set of q points of $\mathscr{B}_k(\Delta)$ containing t_1. By H, there exists $w_2 \in W_k^{\mu}(I_1\Delta)$ with $\lambda_t w_2 = 0$ for all $t \in P_1$, but $\lambda_{t_2} w_2 \neq 0$ for some $t_2 \in \mathscr{B}_{||}(\Delta)$. Now let P_2 be a set of Q points of $\mathscr{B}_k(\Delta)$ containing both t_1 and t_2. By H, there exists $w_3 \in W_k^{\mu}(I_1\Delta)$ with $\lambda_t w_3 = 0$ for all $t \in P_2$, but $\lambda_{t_3} w_3 \neq 0$ for some $t_3 \in \mathscr{B}_k(\Delta)$. Continuing this process, we end up with $Q+1$ splines $w_1, \ldots, w_{Q+1}$ and $Q+1$ points $t_1, \ldots, t_{Q+1}$ such that

$$\lambda_{t_j} w_i = 0, \quad j = 1, \ldots, i-1, \qquad \lambda_i w_i \neq 0$$

holds for $i = 1, \ldots, q+1$. This asserts the linear independence of $w_1, \ldots, w_{Q+1}$, and we have our conclusion. $\square$

Now we discuss $\dim W_2^1(I_1\Delta)$. We can number the vertices of the triangulation Δ in such a way that each pair of consecutive vertices in the list are corners of a common subset in Δ (cf. Fig. 2). For each $i = 1, 2, \ldots, V_{\mathrm{I}}$, let $\tilde{e}_i =$ number of edges attached to the ith vertex but not attached to any of the first $i-1$ vertices in the list and let $\sigma_i = \mathrm{sign}(\tilde{e}_i)$, where

$$\mathrm{sign}(x) = \begin{cases} 1 & x = 2 \\ 0 & x \neq 2 \end{cases}.$$

Then we have

Lemma 5. $\dim W_2^1(I_1\Delta) \leqslant 3E_{\mathrm{I}} - 6(V_{\mathrm{I}} - 1) + \sum_{i=1}^{V_{\mathrm{I}}} \sigma_i$.

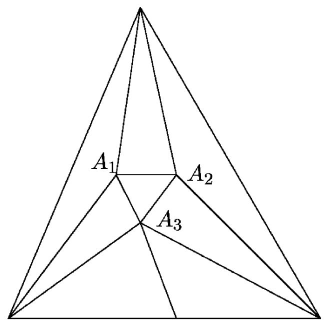

Fig. 2. A triangulation.

Proof. Obviously, the number of edges attached to an interior vertex is not <3.

Suppose the vertices of the partition are $A_1,\dots,A_{V_\mathrm{I}}$ and let $\Omega^1=\cup\{\Omega_i\colon \Omega_i$ has a vertex at $A_1\}$. By Lemma 4, we can find a determining P_1 with $|P_1|=3\tilde{e}_1$ which annihilates any function in $w|_{\Omega^1}$, $w\in W_2^1(I_1\Delta)$. We now continue this process one vertex at a time. In particular, we can add a determining set of $3\tilde{e}_i-6+\mathrm{sign}(\tilde{e}_i)$ to P_{i-1} to get a set P_i which annihilates weak splines on

$$\Omega^i=\Omega^{i-1}\cup\{\Omega_j\colon \Omega_j \text{ has a corner at } A_i\}.$$

After proceeding through all vertices and adding 3 domain points associated with each remaining uncounted edge, we end up with a determining set of $3E_\mathrm{I}-6(V_\mathrm{I}-1)+\sum_{i=1}^{V_\mathrm{I}}\sigma_i$ domain points. This completes the proof. □

Theorem 1. *If* $\tilde{e}_i>2$, $i=1,\dots,V_\mathrm{I}$, *then* $\dim W_2^1(I_1\Delta)=6N-3E_\mathrm{I}$.

Proof. By Lemma 2, $\dim W_2^1(I_1\Delta)\geqslant 6N-3E_\mathrm{I}$. By Lemma 5, $\dim W_2^1(I_1\Delta)\leqslant 3E_\mathrm{I}-6(V_\mathrm{I}-1)$. Using Euler formulation, $E_\mathrm{I}-V_\mathrm{I}+1=N$. Therefore, we have $6N-3E_\mathrm{I}=3E_\mathrm{I}-6(V_\mathrm{I}-1)$. Hence, $\dim W_2^1(I_1\Delta)=6N-3E_\mathrm{I}$. □

4. B splines on type-1 triangulation

Let $R=\{(x,y)\colon 0\leqslant x,y\leqslant 1\}$, $0=x_0<\cdots<x_m=1$ and $0=y_0<\cdots<y_n=1$. Then, the lines $x-x_i=0, 1\leqslant i\leqslant m-1$ and $y-y_j=0$, $0\leqslant j\leqslant n-1$ divide R into mn rectangles which are denoted by

$$R_{ij}=\{(x,y)\colon x_{i-1}\leqslant x\leqslant x_i, y_{j-1}\leqslant y\leqslant y_j\}.$$

Then by drawing in diagonals with positive slopes to the rectangle $[x_i,x_{i+1}]\times[y_j,y_{j+1}]$, we obtain a type-1 triangulation $\Delta_{mn}^{(1)}$ (cf. Fig. 3).

We have

Theorem 2. $\dim W_2^1(I_1\Delta_{mn}^{(1)})=3(mn+m+n)$.

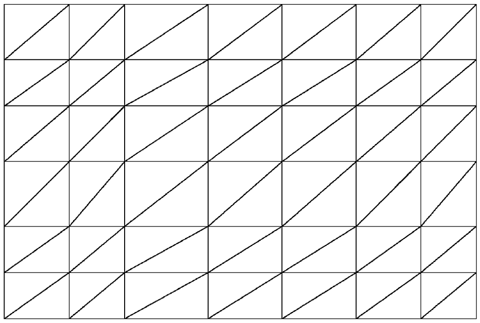

Fig. 3. Type-1 triangulation.

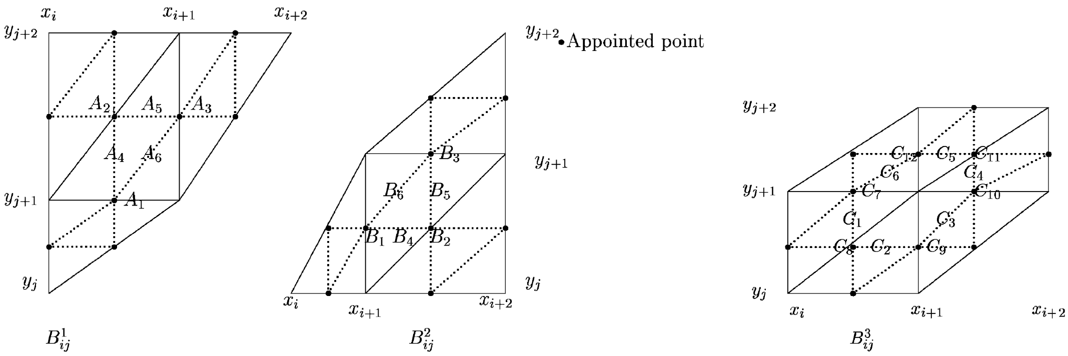

Fig. 4. Several B splines on type 1 triangulation.

Proof. If we put the vertices in lexicographical order, then it is easy to see that $\tilde{e}_i$ is always at least three. Now applying Theorem 1, we obtain the result. $\square$

For the case that all appointed points are the midpoints of grid segments, we find three kinds of B-splines of $W_2^1(I_1\Delta_{mn}^{(1)})$. Fig. 4 shows these B-spline bases.

In Fig. 4 $A_i, B_i, C_j, 1 \leqslant i \leqslant 6, 1 \leqslant j \leqslant 10$ denote the barycentric coordinate and $A_1 = (y_{i+1} - y_i)/(y_{i+2} - y_i)$, $A_2 = \frac{1}{2}$, $A_3 = (x_{i+2} - x_{i+1})/(x_{i+2} - x_i)$, $A_4 = 1$, $A_5 = 1$, $A_6 = 1$, $B_1 = (x_{i+1} - x_i)/(x_{i+2} - x_i)$, $B_2 = \frac{1}{2}$, $B_3 = (y_{i+2} - y_{i+1})/(y_{i+2} - y_i)$, $B_4 = 1$, $B_5 = 1$, $B_6 = 1$, $C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = 1$, $C_7 = (y_{i+2} - y_{i+1})/(y_{i+2} - y_i)$, $C_8 = \frac{1}{2}$, $C_9 = (x_{i+2} - x_{i+1})/(x_{i+2} - x_i)$, $C_{10} = (y_{i+1} - y_i)/(y_{i+2} - y_i)$, $C_{11} = \frac{1}{2}$, $C_{12} = (x_{i+1} - x_i)/(x_{i+2} - x_i)$. All other barycentric coordinates are zero.

Let $S_{ij}^1, S_{ij}^2, S_{ij}^3$ be the three kinds of B-splines, respectively. Let $F_p^{i,j} = \{S_{ij}^p \mid S_{ij}^p$ is not zero identically on $R_{ij}\}$, $F_p = \cup_{i,j} F_p^{i,j}$, and $B = \cup_{p=1,2,3} F_p$, $1 \leqslant p \leqslant 3$.

It is clear that $|B| = 3(nm + n + m) + 1$. Hence the collection B must be linearly dependent on D. We will show a criterion to delete an element from B for getting a local basis of $W_2^1(I_1\Delta_{mn}^{(1)})$.

Lemma 6. *For any given $f \in B$, the elements of $B - f$ are linearly independent.*

Proof. Consider a sub-rectangle of R, say R_0. Let $G = \{g \mid g \in B, g$ is not zero identically on $R_0\}$. It is clear that $|G| = 10$. Denote by $g_i, 1 \leqslant i \leqslant 10$ the elements in G. According to

$$\sum_{1 \leqslant i \leqslant 10} a_i g_i|_{R_0} \equiv 0, \tag{7}$$

we get a linear system. The rank of the coefficient matrix of the linear system is 9, i.e., if a B_j is deleted, then $a_i = 0, 1 \leqslant i \leqslant 10$. By Fig. 3, it is easy to prove that the coefficient of B spline on adjacent subrectangle is also 0. Hence for any $f \in B$, the elements of $B - f$ are linearly independent. $\square$

By $\dim W_2^1(I_1 \varDelta_{mn}^{(1)})$ and Lemma 6, we have

Theorem 3. *For any given $f \in B$, $B - f$ is a basis of $W_2^1(I_1 \varDelta_{mn}^{(1)})$.*

Theorem 4. (1) *For all $(x, y) \in R_{ij}$*

$$\sum_{p=1}^{2} \sum_{S_{ij}^p \in F_p^{i,j}} S_{ij}^p(x, y) \equiv 1, \quad \sum_{S_{ij}^3 \in F_3} S_{ij}^3(x, y) \equiv 1.$$

(2) *$S_{ij}^1(x, y)$, $S_{ij}^2(x, y)$, $S_{ij}^3(x, y)$ are defined uniquely inside their supports, and every function in $W_2^1(I_1 \varDelta_{mn}^1)$ supported by the support of $S_{ij}^p(x, y)$, $(1 \leqslant p \leqslant 3)$ is a constant multiple of $S_{ij}^p(x, y)$ $(1 \leqslant p \leqslant 3)$.*

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[1–4,6,7,9]

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